Model Reduction for Delay Differential Equations with Guaranteed Stability and Error Bound

N. van de Wouw [a,b,1], W. Michiels [c], B. Besselink [d]

[a] Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands

[b] Department of Civil, Environmental & Geo- Engineering, University of Minnesota, Minneapolis, U.S.A.

[c] Department of Computer Science, KU Leuven, Leuven, Belgium

[d] School of Electrical Engineering, KTH, Stockholm, Sweden

Abstract

In this paper, a structure-preserving model reduction approach for a class of delay differential equations is proposed. Benefits of this approach are, firstly, the fact that the delay nature of the system is preserved after reduction, secondly, that input-output stability properties are preserved and, thirdly, that a computable error bound reflecting the accuracy of the reduction is provided. These results are applicable to large-scale linear delay differential equations with constant delays, but also extensions to a class of nonlinear delay differential equations with time-varying delays are presented. The effectiveness of the results is evidenced by means of an illustrative example.

Key words: Model reduction, delay differential equations, stability, error bound, nonlinearity, time-varying delays.

1 Introduction

Complex dynamical system models in terms of delay differential equations appear naturally in a wide variety of problems in for example engineering, biology and control theory [1, 11, 21, 35, 40]. In support of the dynamic analysis, optimization or controller design for such systems, we often desire to employ methods for model complexity reduction. Model order reduction is a tool for the order reduction of high-order dynamical systems in pursuit of complexity reduction. A wide range of results are available for the model order reduction of models in terms of ordinary differential equations, see e.g. [2, 4, 9, 10, 14, 16, 22].

Also for delay differential equations (DDEs) different approaches for model reduction are available, albeit to a more limited extent. Methods for the finite-dimensional approximation of delay systems through rational approximations have been proposed in [31, 32], see also [20]. Recently, a technique based on the dominant pole algorithm has been proposed to obtain a rational approximation of a input-output transfer function representing second-order delay differential equations [37]. A Krylov-based model reduction approach leading to finite-dimensional (delay-free) model approximations has been proposed in [34]. In [24], Krylov methods for infinite-dimensional systems, applicable to delay systems, have been proposed also leading to finite-dimensional approximations. The above methods have the common property that the resulting models are of a finite-dimensional nature; hence the inherent delay nature of the original system is lost.

In this paper, we aim at constructing reduced-order models which preserve the delay nature of the system dynamics (i.e. the reduced-order model is also a delay differential equation, though of a reduced order). The desire to preserve the delay nature in the reduced-order model is motivated by, firstly, the fact that, for a given order of the reduced model, a reduced model in the form of a delay differential equation is in general more accurate than a reduced model in the form of a delay free system, see e.g. [37], and, secondly, the fact that by preserving the delay nature also related system properties (such as e.g. the infinite-dimensional system character

Email addresses: n.v.d.wouw@tue.nl (N. van de Wouw), Wim.Michiels@cs.kuleuven.be (W. Michiels), bart.besselink@ee.kth.se (B. Besselink).
[1] Corresponding author N. van de Wouw. Tel. +31 40 2473358. Fax +31 40 2461418.

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and the infinite number of eigenvalues) are preserved. Such structure-preserving model reduction techniques for delay differential equations, yielding reduced-order delay models, are needed as, on the one hand, powerful simulation and controller synthesis techniques for such systems have become available in the recent past [6, 21, 35, 38], while, on the other hand, the main bottleneck of these methods is that in most cases they require the order of the delay differential equation to be moderate. In [5], interpolatory projection methods based have been proposed, which are also applicable to delay systems and preserve the delay nature in the reduced-order model. In [27], a structure preserving model reduction technique for delay differential equations has been proposed, which extends the notion of position balancing from second-order systems to time-delay systems and relies on solving delay Lyapunov equations [28].

In this paper, we propose a structure-preserving model order reduction strategy for a class of delay differential equations, based on balancing techniques, which, firstly, preserves the delay nature of the model, secondly, guarantees the preservation of both internal and input-output stability properties and, thirdly, comes with a computable error bound on the reduced-order model. We note that the latter two aspects (stability preservation and an error bound) are lacking in the existing results in the literature mentioned above. Error bounds have been proposed for finite-dimensional rational approximations, see [20]. Moreover, error bounds and the preservation of stability is also guaranteed in the works [29,42], in which an H_∞ model reduction approach for linear time-delay systems has been proposed.

The benefits of the approach proposed in the current paper in comparison with the approach in [29, 42] is twofold. Firstly, by the grace of the fact that we employ balancing-type techniques as a basis, which use the solution to two algebraic Lyapunov equations, the approach proposed here is applicable to systems up to order $O(10^3)$ using standard (Bartels-Stewart) algorithms and to systems up to order $O(10^6)$ using tailored algorithms, see e.g. [7]. On the other hand, the approach in [29,42] of reformulating the model reduction problem as a H_∞-norm minimization problem of the 'error system', induced by the reduction, leads to an (non-convex) optimization problem constrained by a set of matrix inequalities. The latter fact makes such an approach more computationally complex and hence obstructs applicability to systems of high order. Secondly, we propose a natural approach of decomposing the delay system dynamics in terms a feedback interconnection between a finite-dimensional linear part and a delay-operator part. This approach is natural in many applications, in which the delay only affects certain outputs, see e.g. models for high-speed milling processes [1, 12, 26] and drilling processes [17,18]. Moreover, such a decomposition allows to employ incremental $\mathcal{L}_2$-gain properties of the systems in the feedback interconnection to guarantee the preservation of stability and to provide an error bound. The latter analysis strategy is also instrumental in supporting the extension of the model reduction approach to systems with nonlinearities and (uncertain) time-varying delays. Finally, we provide an expression for an a priori error bound depending on 1) the properties of the high-order system, 2) the delay and 3) the order of the reduced-order system.

The structure of the paper is as follows. Section 2 specifies in detail the problem formulation and the class of delay systems considered. Next, in Section 3 the model reduction approach is introduced as applicable to a class of linear delay differential equations with constant delays. Section 4 presents the results on the preservation of stability properties and a bound on the reduction error. Moreover, in this section also the extension to nonlinear systems with time-varying delays is highlighted. Finally, Section 5 presents an illustrative example and Section 6 presents concluding remarks.

Notation. The field of real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, $|x|^2 = x^T x$. The space $\mathcal{L}_2^n$ consists of all functions $x : [0, \infty) \to \mathbb{R}^n$ which are bounded using the norm $\|x\|_2^2 := \int_0^\infty |x(t)|^2 dt$.

2 Problem Formulation

Consider a generic class of linear delay differential equations (with point-wise delay) that can be formulated in the following form:

$$\Sigma : \begin{cases} \dot{x}(t) = \bar{A}_0 x(t) + \bar{A}_1 x(t-\tau) + Bu(t), \\ y(t) = C_y x(t) + D_{yu} u(t) \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^p$. Alternatively, the dynamics in (1) can be written in the following form, to be used in the remainder of this paper:

$$\Sigma : \begin{cases} \dot{x}(t) = A_0 x(t) + A_1(x(t) - x(t-\tau)) + Bu(t), \\ y(t) = C_y x(t) + D_{yu} u(t) \end{cases} \tag{2}$$

with $A_0 = \bar{A}_0 + \bar{A}_1$ and $A_1 = -\bar{A}_1$.

We study the problem of model reduction for delay differential equations of the form (2) and later comment on extensions to certain classes of nonlinear systems and the case of (uncertain) time-varying delays. Let us explicate what we mean by model reduction for a delay differential equation as in (2). Hereto, we recall the fact that the model in (2) is infinite-dimensional, i.e. the initial condition for system (2) is the function segment $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ with $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to $\mathbb{R}^n$. In fact, we aim to preserve the infinite-dimensional

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nature of the system in the model reduction approach to be proposed. Still, we can speak of the order of the delay differential equation (2) in terms of the number of equations in the first equality in (2), which in this case is n. Now, we aim at constructing a reduced-order model in terms of a linear delay differential equation of order $\hat{n}$ (i.e. with 'state' $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$) such that,

- the reduced-order model is also a delay differential equation similar in form to (2), i.e. the delay-nature of the system is preserved;
- $\hat{n} < n$, i.e. model (order) reduction is achieved;
- if (2) is asymptotically stable (for $u = 0$) and hence finite $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y), then the reduced-order model is also asymptotically stable asymptotically stable (for $u = 0$) and $\mathcal{L}_2$-gain stable with respect to the same input/output pair $(u, \hat{y})$, where $\hat{y}$ is the output of the reduced-order system;
- there exists a computable error bound reflecting the accuracy of the reduction.

Clearly, in the above problem statement we aim at the preservation of asymptotic stability for zero inputs [2] and $\mathcal{L}_2$-gain stability with respect to the input/output pair (u, y), the latter of which is defined below (see also [15]).

Definition 1
System (2) is called $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y) with finite gain γ if for solutions of (2) corresponding to the zero initial condition ($\phi = 0$) it holds that $\|y\|_2 \leq \gamma \|u\|_2$.

Remark 1
We foresee that the results in this paper can be extended towards systems of the form (2) with multiple delays. For the sake of transparency and to alleviate the burden of notation, we do not pursue this extension explicitly in this paper.

3 Model Reduction Approach

In support of the pursuit of the model reduction of system Σ in (2), let us transform this system into a feedback interconnection of a finite-dimensional linear system Σ_1 and an operator Σ_2 related to the delay (we will denote this feedback interconnection by (Σ_1, Σ_2)):

$$\Sigma_1 : \begin{cases} \dot{x}(t) = A_0 x(t) + B_v v(t) + B_u u(t), \\ w(t) = C_w x(t) + D_{wv} v(t) + D_{wu} u(t), \\ y(t) = C_y x(t) + D_{yu} u(t), \end{cases} \tag{3}$$

$$\Sigma_2 : v(t) = \int_{t-\tau}^{t} w(s) ds, \tag{4}$$

[2] For a definition of asymptotic stability for functional differential equations, we refer to [21, 23].

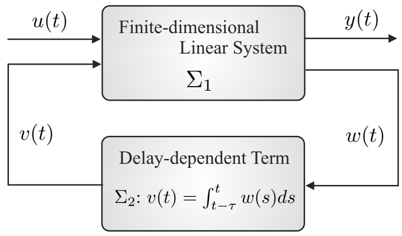

Figure 1. Schematic of system decomposition in (3), (4).

where $v(t), w(t) \in \mathbb{R}^q$ and we employed a (rank revealing) decomposition of the matrix A_1 in (2) in the form $A_1 = B_v C_z$. In other words, the latter decomposition should be performed such that B_v has a minimum number of columns in order to make the model reduction pursued hereafter most effective. Moreover, in (3) we defined $B_u := B$, $C_w := C_z A_0$, $D_{wv} := C_z B_v$ and $D_{wu} := C_z B$. In interpreting how (3), (4) represents (2), it helps to realize that $v(t) = C_z(x(t) - x(t - \tau))$ and $w(t) = \dot{z}(t)$ with $z(t) = C_z x(t)$.

In many engineering applications in which models are formulated as delay differential equations, such as e.g. models for high-speed milling processes [1, 12, 26] and drilling processes [17, 18], the matrix A_1 indeed has low rank. Namely, in such models the high-order x-related dynamics typically corresponds to models of the structural dynamics of the spindle-tool dynamics in high speed milling or the drill-string dynamics in drilling, while the delay-related terms relate to localized cutting processes depending on low-dimensional variables. Similarly, in the context of boundary control of partial differential equations, feedback delays affect control inputs localized at the boundary also leading to models with the matrix A_1 having low rank.

The system decomposition as a feedback interconnection of a finite-dimensional linear system and a delay-dependent term, see (3), (4), is schematically depicted in Figure 1. Clearly, with such decomposition we pursue a delay-dependent approach towards the analysis of the delay system involved, see e.g. [21]. Moreover, the form of the system decomposition in (3), (4) naturally supports a model reduction strategy in which the order of Σ_1 is reduced, while Σ_2 is left unchanged. In this way, we meet the objectives, as put forward in Section 2, of achieving order reduction while preserving the delay nature of the system. In particular, we show in Section 3.1 that with a particular reduction approach a reformulation of the reduced system as a DDE is possible.

Let ust adopt the following assumption on system (3).

Assumption 1
Σ_1 is asymptotically stable (i.e. A_0 is Hurwitz).

Remark 2
Note that, due to the asymptotic stability of Σ_1 (Assumption 1), there exist input-output operators $\mathcal{F}_y$:

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$\mathcal{L}_2^p \times \mathcal{L}_2^q \to \mathcal{L}_2^m$ and $\mathcal{F}_w : \mathcal{L}_2^p \times \mathcal{L}_2^q \to \mathcal{L}_2^q$ defined as $y = \mathcal{F}_y(u,v)$ and $w = \mathcal{F}_w(u,v)$, respectively. These operators generate the outputs y and w of the finite-dimensional linear system Σ_1 for given inputs u and v and zero initial condition $x(0) = 0$. Linearity and asymptotic stability of Σ_1 together imply a bounded incremental $\mathcal{L}_2$ gain property, such that the above input-output operators satisfy

$$\begin{aligned} \|\mathcal{F}_i(u_1,v_1) - \mathcal{F}_i(u_2,v_2)\|_2 \leq &\gamma_{iu}\|u_1 - u_2\|_2 \\ &+ \gamma_{iv}\|v_1 - v_2\|_2, \end{aligned} \tag{5}$$

for all $u_1,\, u_2 \in \mathcal{L}_2^p$, $v_1,\, v_2 \in \mathcal{L}_2^q$, and some bounded γ_{iu}, $\gamma_{iv} \geq 0$ with $i \in \{y,w\}$. Due to linearity, the incremental $\mathcal{L}_2$ gain is equivalent to the (non-incremental) $\mathcal{L}_2$ gain, such that the gains γ_{ij} in (5) can be chosen as the H_∞-norm of the corresponding transfer functions.

Later, we will use the following lemma on an incremental gain property of the operator Σ_2.

Lemma 1
The operator Σ_2 satisfies the following incremental gain property:

$$\|v_2 - v_1\|_2 \leq \tau\|w_2 - w_1\|_2, \quad \forall\, w_1,\, w_2 \in \mathcal{L}_2^q. \tag{6}$$

Proof The proof for the non-incremental version of (6), i.e. $\|v\|_2 \leq \tau\|w\|_2$, for all w, is given in [15, 33]. Due to linearity of the operator Σ_2 this fact also implies the validity of the incremental gain property in (6). □

Let us now adopt the following assumption on the feedback interconnection (Σ_1, Σ_2) given by (3), (4).

Assumption 2
The feedback interconnection (Σ_1, Σ_2) satisfies the small-gain condition

$$\gamma_{wv}\tau < 1. \tag{7}$$

Remark 3
Due to the asymptotic stability of Σ_1 (Assumption 1), γ_{wv} always exists (i.e. is bounded) and hence (7) can always be satisfied for small enough delay τ.

Lemma 2
Consider system (3), (4) satisfying Assumptions 1 and 2. Then the feedback interconnection (Σ_1, Σ_2) is

- *$\mathcal{L}_2$ gain stable with respect to the input/output pair (u, y);*
- *asymptotically stable for $u = 0$.*

Proof Under Assumption 1, there exist bounded γ_{wu} and γ_{wv} such that $\|w\|_2 \leq \gamma_{wu}\|u\|_2 + \gamma_{wv}\|v\|_2$. Using (7) and the non-incremental version of Lemma 1, we conclude that

$$\|w\|_2 \leq \frac{\gamma_{wu}}{1 - \gamma_{wv}\tau}\|u\|_2. \tag{8}$$

Using (8) and the non-incremental version of Lemma 1 in $\|y\|_2 \leq \gamma_{yu}\|u\|_2 + \gamma_{yv}\|v\|_2$ gives

$$\begin{aligned} \|y\|_2 &\leq \gamma_{yu}\|u\|_2 + \gamma_{yv}\tau\|w\|_2 \\ &\leq \left(\gamma_{yu} + \frac{\gamma_{yv}\tau\gamma_{wv}}{1 - \gamma_{wv}\tau}\right)\|u\|_2, \end{aligned} \tag{9}$$

which shows that (Σ_1, Σ_2) is $\mathcal{L}_2$ gain stable with respect to the input/output pair (u, y). Now, using the fact that system Σ_1 is an asymptotically stable linear time-invariant system, Σ_2 has a finite impulse response and the feedback interconnection (Σ_1, Σ_2) satisfies a small gain condition, we can conclude that (Σ_1, Σ_2) is also asymptotically stable for $u = 0$ (see also [25, 41]). This completes the proof. □

In pursuing model reduction of (3), (4), we construct a reduced-order model $\hat{\Sigma}_1$ for the linear finite-dimensional system Σ_1 in the following form:

$$\hat{\Sigma}_1 : \begin{cases} \dot{\hat{x}}(t) = \hat{A}_0\hat{x}(t) + \hat{B}_v\hat{v}(t) + \hat{B}_u u(t), \\ \hat{w}(t) = \hat{C}_w\hat{x}(t) + \hat{D}_{wv}\hat{v}(t) + \hat{D}_{wu}u(t), \\ \hat{y}(t) = \hat{C}_y\hat{x}(t) + \hat{D}_{yv}\hat{v}(t) + \hat{D}_{yu}u(t) \end{cases} \tag{10}$$

with $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$ and $\hat{n} < n$. For an efficient reduction of the system in (3) to the system in (10), the number of inputs and outputs should be small. For approaches based on balanced truncation, this can be understood from the fact that in such a case the decay rate of the Hankel singular values is fast [3]. In (3), the number of inputs is determined by the dimension of $u(t)$ and the dimension of $v(t)$, the latter of which stems from a feedback interconnection interpretation of the delayed term, see Figure 1. Hence, it is important to keep the size of $v(t)$ (and $w(t)$) as small as possible. This can be done by starting from a rank revealing decomposition of A_1, i.e., such that the dimension of $v(t)$ is equal to $\mathrm{rank}(A_1)$.

Let us adopt the following assumption on the reduced-order linear system $\hat{\Sigma}_1$.

Assumption 3
- *$\hat{\Sigma}_1$ is asymptotically stable;*
- *An (incremental) error bound on reduction of the linear subsystem exists of the form*

$$\begin{aligned} \|\mathcal{E}_i(u_1,v_1) - \mathcal{E}_i(u_2,v_2)\|_2 \leq &\epsilon_{iu}\|u_1 - u_2\|_2 \\ &+ \epsilon_{iv}\|v_1 - v_2\|_2, \end{aligned} \tag{11}$$

for all $u_1,\, u_2 \in \mathcal{L}_2^p$, $v_1,\, v_2 \in \mathcal{L}_2^q$, with $\epsilon_{iu},\, \epsilon_{iv} \geq 0$ and $i \in \{y,w\}$. In (11), $\mathcal{E}_i := \mathcal{F}_i - \hat{\mathcal{F}}_i$, $i \in \{y,w\}$,

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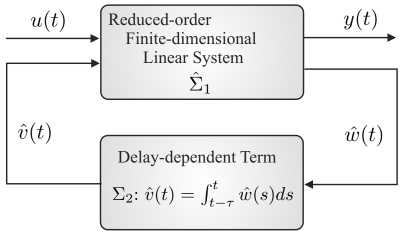

Figure 2. Schematic of reduced-order system decomposition in (12), (13).

denotes the error operator with $\hat{\mathcal{F}}_i : \mathcal{L}_2^p \times \mathcal{L}_2^q \to \mathcal{L}_2^{\{m,q\}}$ *the input-output operators of the reduced-order linear subsystem* $\hat{\Sigma}_1$ *for zero initial condition, which exist by the grace of asymptotic stability and linearity.*

If we employ balanced truncation [36], optimal Hankel norm approximation [19], or balanced residualization [3], then the resulting reduced-order linear system is of the form $\hat{\Sigma}_1$ and satisfies Assumption 3. Note in this respect that the incremental error bound in (11) is, due to linearity, directly implied by an ordinary (i.e. non-incremental) error bound. In Section 3.1, we show that if balanced residualization is used to reduce Σ_1, then the delay-structure of the original system can be preserved in the reduced-order system.

3.1 Formulation of the reduced-order model as a delay differential equation

The reduced-order model $\hat{\Sigma}$ is now given by the feedback interconnection of $\hat{\Sigma}_1$ and Σ_2, denoted by $(\hat{\Sigma}_1, \Sigma_2)$, where Σ_2 relates $\hat{v}$ to $\hat{w}$ according to $\hat{v}(t) = \int_{t-\tau}^{t} \hat{w}(s)ds$, i.e. the dynamics of $\hat{\Sigma}$ can be formulated as follows:

$$\hat{\Sigma}_1 : \begin{cases} \dot{\hat{x}}(t) = \hat{A}_0\hat{x}(t) + \hat{B}_v\hat{v}(t) + \hat{B}_u u(t), \\ \hat{w}(t) = \hat{C}_w\hat{x}(t) + \hat{D}_{wv}\hat{v}(t) + \hat{D}_{wu}u(t), \\ \hat{y}(t) = \hat{C}_y\hat{x}(t) + \hat{D}_{yv}\hat{v}(t) + \hat{D}_{yu}u(t) \end{cases} \quad (12)$$

with

$$\Sigma_2 : \hat{v}(t) = \int_{t-\tau}^{t} \hat{w}(s)ds. \quad (13)$$

Figure 2 depicts a schematic of this reduced-order system decomposition. Now, we consider the natural desire to formulate the reduced-order system as a delay differential equation in a form similar to that of the original system in (2).

[3] By balanced residualization, we indicate the singular perturbation approximation of balanced realizations as proposed in [13, 30].

Let us now present an approach for achieving such structure-preserving model reduction. Hereto, we assume, without loss of generality, that system Σ_1 in (3) is a balanced realization [4]. Moreover, we partition the matrices A_0, B_v, B_u, C_y and C_w as follows:

$$\begin{aligned} A_0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ B_v = \begin{bmatrix} B_{v1} \\ B_{v2} \end{bmatrix}, \ B_u = \begin{bmatrix} B_{u1} \\ B_{u2} \end{bmatrix}, \\ C_y &= \begin{bmatrix} C_{y1} & C_{y2} \end{bmatrix}, \ C_w = \begin{bmatrix} C_{w1} & C_{w2} \end{bmatrix}, \end{aligned} \quad (14)$$

in accordance with the partitioning of the state $x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$ with $x_1 \in \mathbb{R}^{\hat{n}}$, $x_2 \in \mathbb{R}^{n-\hat{n}}$.

Let us now employ a singular perturbation approach (i.e. based on balanced residualization [13, 30]) to obtain the following reduced-order system $\hat{\Sigma}_1^{br}$ for system Σ_1:

$$\hat{\Sigma}_1^{br} : \begin{cases} \dot{\hat{x}}(t) = (A_{11} - A_{12}A_{22}^{-1}A_{21})\hat{x}(t) \\ \qquad + (B_{v1} - A_{12}A_{22}^{-1}B_{v2})\hat{v}(t) \\ \qquad + (B_{u1} - A_{12}A_{22}^{-1}B_{u2})u(t), \\ \hat{w}(t) = (C_{w1} - C_{w2}A_{22}^{-1}A_{21})\hat{x}(t) \\ \qquad + (D_{wv} - C_{w2}A_{22}^{-1}B_{v2})\hat{v}(t) \\ \qquad + (D_{wu} - C_{w2}A_{22}^{-1}B_{u2})u(t), \\ \hat{y}(t) = (C_{y1} - C_{y2}A_{22}^{-1}A_{21})\hat{x}(t) \\ \qquad - C_{y2}A_{22}^{-1}B_{v2}\hat{v}(t) \\ \qquad + (D_{yu} - C_{y2}A_{22}^{-1}B_{u2})u(t) \end{cases} \quad (15)$$

with $\hat{x}$ approximating x_1. Note that $\hat{\Sigma}_1^{br}$ above is of the form of $\hat{\Sigma}_1$ as in (10), albeit with a particular structure for the system matrices due to the particular usage of balanced residualization as a reduction strategy. The following proposition shows that the reduced-order system $\hat{\Sigma}^{br} = (\hat{\Sigma}_1^{br}, \Sigma_2)$ with $\hat{\Sigma}_1^{br}$ as in (15) can indeed be written in terms of a delay differential equation. A key ingredient of this proposition is in showing that there exists a matrix $\hat{C}_z$ such that $\hat{w}$ as in (15) satisfies $\hat{w} = \hat{C}_z\dot{\hat{x}}$. The latter fact can then be used to show that $\hat{v}$ satisfying (13) can be expressed as $\hat{v} = \int_{t-\tau}^{t} \hat{C}_z\dot{\hat{x}}(s)ds = \hat{C}_z(\hat{x}(t) - \hat{x}(t-\tau))$, thereby reformulating the distributed delay term in (13) in terms of an expression involving a point-wise delay.

Proposition 1
Consider the reduced-order system $\hat{\Sigma}^{br} = (\hat{\Sigma}_1^{br}, \Sigma_2)$. *These system dynamics can be reformulated in terms of*

[4] Proposition 1 below holds for systems Σ_1 in arbitrary coordinates.

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the following delay differential equation

$$\begin{aligned}\dot{\hat{x}}(t) &= \hat{A}_0\hat{x}(t) + \hat{A}_1(\hat{x}(t) - \hat{x}(t-\tau)) + \hat{B}_u u(t),\\ \hat{y}(t) &= \hat{C}_y\hat{x}(t) + \hat{C}_{yx}(\hat{x}(t) - \hat{x}(t-\tau)) + \hat{D}_{yu}u(t)\end{aligned} \tag{16}$$

with

$$\begin{aligned}\hat{A}_0 &:= A_{11} - A_{12}A_{22}^{-1}A_{21},\ \ \hat{A}_1 := \hat{B}_v\hat{C}_z,\\ \hat{B}_v &:= B_{v1} - A_{12}A_{22}^{-1}B_{v2},\ \ \hat{B}_u := B_{u1} - A_{12}A_{22}^{-1}B_{u2},\\ \hat{C}_{yx} &:= \hat{C}_{yz}\hat{C}_z,\ \ \hat{C}_z := C_{z1},\ \ \hat{C}_{yz} := -C_{y2}A_{22}^{-1}B_{v2},\\ \hat{C}_y &:= C_{y1} - C_{y2}A_{22}^{-1}A_{21},\ \ \hat{D}_{yu} := D_{yu} - C_{y2}A_{22}^{-1}B_{u2},\end{aligned} \tag{17}$$

where $C_z = \begin{bmatrix} C_{z1} & C_{z2}\end{bmatrix}$ *is a partitioning in accordance with the partitioning in* (14).

Proof We aim to prove that system $\hat{\Sigma}^{br} = (\hat{\Sigma}_1^{br}, \Sigma_2)$, with $\hat{\Sigma}_1^{br}$ as in (15), can be written as (16), (17). To enable such reformulation of the reduced-order system dynamics we take $\hat{A}_0$ and $\hat{B}_u$ compliant with (17). For the same purpose, we also require that there exists a matrix $\hat{C}_z$ such that $\hat{w}(t) = \hat{C}_z\dot{\hat{x}}(t)$ holds. If such a matrix $\hat{C}_z$ indeed exists, then $\hat{v}(t)$ can be written as follows:

$$\begin{aligned}\hat{v}(t) &= \int_{t-\tau}^{t}\hat{w}(s)ds = \hat{C}_z\int_{t-\tau}^{t}\dot{\hat{x}}(s)ds\\ &= \hat{C}_z(\hat{x}(t) - \hat{x}(t-\tau)),\end{aligned} \tag{18}$$

which enables the system reformulation as in the proposition with $\hat{B}_v := B_{v1} - A_{12}A_{22}^{-1}B_{v2}$, which is satisfied due to (17), and $\hat{A}_1 = \hat{B}_v\hat{C}_z$ and $\hat{C}_{yx} = \hat{C}_{yz}\hat{C}_z$.

In order to facilitate the existence of such a matrix $\hat{C}_z$, it should hold that $\hat{w}(t) = \hat{C}_z\dot{\hat{x}}(t)$, i.e.

$$\begin{aligned}\hat{C}_z\hat{A}_0 &= C_{w1} - C_{w2}A_{22}^{-1}A_{21},\\ \hat{C}_z\hat{B}_v &= D_{wv} - C_{w2}A_{22}^{-1}B_{v2},\\ \hat{C}_z\hat{B}_u &= D_{wu} - C_{w2}A_{22}^{-1}B_{u2}.\end{aligned} \tag{19}$$

Using the fact that (see matrix definitions after (4))

$$\begin{aligned}C_w &= \begin{bmatrix} C_{w1} & C_{w2}\end{bmatrix} = \begin{bmatrix} C_{z1} & C_{z2}\end{bmatrix}\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\\ &= \begin{bmatrix} C_{z1}A_{11} + C_{z2}A_{21} & C_{z1}A_{12} + C_{z2}A_{22}\end{bmatrix},\end{aligned} \tag{20}$$

the first condition in (19) can be written as

$$\begin{aligned}&\hat{C}_z\hat{A}_0 = C_{w1} - C_{w2}A_{22}^{-1}A_{21}\\ \Leftrightarrow\ &\hat{C}_z\hat{A}_0 = C_{z1}A_{11} + C_{z2}A_{21}\\ &\qquad - (C_{z1}A_{12} + C_{z2}A_{22})A_{22}^{-1}A_{21}\\ \Leftrightarrow\ &\hat{C}_z\hat{A}_0 = C_{z1}(A_{11} - A_{12}A_{22}^{-1}A_{21}),\end{aligned} \tag{21}$$

which is satisfied by the grace of (17) in the proposition.

By decomposing $B_v = \begin{bmatrix} B_{v1}^T & B_{v2}^T\end{bmatrix}^T$ and using the fact that (see matrix definitions after (4)) $D_{wv} = \begin{bmatrix} C_{z1} & C_{z2}\end{bmatrix}\begin{bmatrix} B_{v1}^T & B_{v2}^T\end{bmatrix}^T = C_{z1}B_{v1} + C_{z2}B_{v2}$, the second condition in (19) can be written as

$$\begin{aligned}&\hat{C}_z\hat{B}_v = D_{wv} - C_{w2}A_{22}^{-1}B_{v2}\\ \Leftrightarrow\ &\hat{C}_z\hat{B}_v = C_{z1}B_{v1} + C_{z2}B_{v2}\\ &\qquad - (C_{z1}A_{12} + C_{z2}A_{22})A_{22}^{-1}B_{v2}\\ \Leftrightarrow\ &\hat{C}_z\hat{B}_v = C_{z1}(B_{v1} - A_{12}A_{22}^{-1}B_{v2}),\end{aligned} \tag{22}$$

which is satisfied by the grace of (17) in the proposition.

Using the fact that (see matrix definitions after (4)) $D_{wu} = C_{z1}B_{u1} + C_{z2}B_{u2}$, the third condition in (19) can be written as

$$\begin{aligned}&\hat{C}_z\hat{B}_u = D_{wu} - C_{w2}A_{22}^{-1}B_{u2}\\ \Leftrightarrow\ &\hat{C}_z\hat{B}_u = C_{z1}B_{u1} + C_{z2}B_{u2}\\ &\qquad - (C_{z1}A_{12} + C_{z2}A_{22})A_{22}^{-1}B_{u2}\\ \Leftrightarrow\ &\hat{C}_z\hat{B}_u = C_{z1}(B_{u1} - A_{12}A_{22}^{-1}B_{u2}),\end{aligned} \tag{23}$$

which is satisfied by the grace of (17) in the proposition. Now, we have shown that there indeed exists a matrix $\hat{C}_z$ that satisfies $\hat{w} = \hat{C}_z\dot{\hat{x}}$.

Finally, in order to show that the output equation for $\hat{y}$ can indeed be written as in (16), we write the output equation for $\hat{y}$ as in (15) as follows:

$$\begin{aligned}&\hat{y}(t) = (C_{y1} - C_{y2}A_{22}^{-1}A_{21})\hat{x}(t) - C_{y2}A_{22}^{-1}B_{v2}\hat{v}(t)\\ &\qquad + (D_{yu} - C_{y2}A_{22}^{-1}B_{u2})u(t)\\ \Leftrightarrow\ &\hat{y}(t) = (C_{y1} - C_{y2}A_{22}^{-1}A_{21})\hat{x}(t)\\ &\qquad - C_{y2}A_{22}^{-1}B_{v2}\hat{C}_z(\hat{x}(t) - \hat{x}(t-\tau))\\ &\qquad + (D_{yu} - C_{y2}A_{22}^{-1}B_{u2})u(t),\end{aligned} \tag{24}$$

where we used once more that $\hat{v}(t) = \int_{t-\tau}^{t}\hat{w}(s)ds = \int_{t-\tau}^{t}\hat{C}_z\dot{\hat{x}}(s)ds = \hat{C}_z(\hat{x}(t) - \hat{x}(t-\tau))$. Comparison of (24) with (16) shows that the output equation for $\hat{y}$ can indeed be written as in (16) by the grace of (17) in the proposition. This completes the proof. □

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Note that the reduced-order system (16) has now been formulated in a form similar to that of the original system (2). The only potential difference is the fact that a delay may appear in the output equation for $\hat{y}$ (depending on whether $\hat{C}_{yz}$ in (17) is non-zero or not).

Remark 4
It can be shown that if a reduced-order model $\hat{\Sigma}_1$ is constructed by moment matching techniques [2] that ensure matching of the moment at $s = 0$, then the resulting reduced-order model $\hat{\Sigma} = (\hat{\Sigma}_1, \Sigma_2)$ can also be reformulated in terms of a delay differential equation of the form (16). *However, since Assumption 3 does in general not hold for such moment matching techniques, while it does for balanced residualization, and Assumption 3 will prove to be essential in proving both stability and and error bound (see the next section), we have limited the formulation of Proposition 1 to the case of balanced residualization. Finally, we note that balanced residualization also matches the moment at $s = 0$.*

4 Stability Analysis and Error bound

The following result provides conditions under which, firstly, the reduced-order system inherits certain stability properties from the original system, and, secondly, an error bound can be computed reflecting the accuracy of the reduction.

Theorem 1
Suppose the system (3), (4) *satisfies Assumption 1. Let $\hat{\Sigma}_1$ in* (10) *be a reduced-order linear system satisfying Assumption 3. Then, the following statements hold:*

(1) The reduced-order system $(\hat{\Sigma}_1, \Sigma_2)$ given by (10), (4) *is $\mathcal{L}_2$ stable with respect to the input/output pair (u, y) and asymptotically stable for $u = 0$ if*

$$(\gamma_{wv} + \epsilon_{wv})\tau < 1; \tag{25}$$

(2) Suppose (25) *is satisfied. Then, the output error $\delta y := y - \hat{y}$ is bounded as $\|\delta y\|_2 \leq \epsilon \|u\|_2$ with*

$$\begin{aligned}\epsilon = \epsilon_{yu} &+ \frac{\epsilon_{yv}\tau\gamma_{wu}}{1-\gamma_{wv}\tau} \\ &+ \frac{(\gamma_{yv}+\epsilon_{yv})\tau}{1-(\gamma_{wv}+\epsilon_{wv})\tau}\left(\epsilon_{wu} + \frac{\epsilon_{wv}\tau\gamma_{wu}}{1-\gamma_{wv}\tau}\right).\end{aligned} \tag{26}$$

Proof Inspired by the work in [8], statements (1) and (2) are proven separately below.
Statement (1): Lemma 2 can be employed to show that if $\hat{\gamma}_{wv}\tau < 1$, then statement (1) of the theorem is valid. Note that $\hat{\gamma}_{wv}$ denotes the $\mathcal{L}_2$-gain of system $\hat{\Sigma}_1$ from input w to output v, which is bounded by the grace of asymptotic stability of $\hat{\Sigma}_1$ (Assumption 3). However, the gain $\hat{\gamma}_{wv}$ is not known a priori (i.e. before actually performing the reduction). Still, we can obtain an upper bound for $\hat{\gamma}_{wv}$ as follows. By the triangle inequality, we have that $\|\hat{w}\|_2 \leq \|w\|_2 + \|w - \hat{w}\|_2$, which implies that $\|\hat{w}\|_2 \leq \gamma_{wv}\|v\|_2 + \gamma_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2 + \epsilon_{wu}\|u\|_2 \Rightarrow \|\hat{w}\|_2 \leq (\gamma_{wv} + \epsilon_{wv})\|v\|_2 + (\epsilon_{wu} + \gamma_{wu})\|u\|_2$, where we used (11) for $i = w$. Clearly, $(\gamma_{wv} + \epsilon_{wv})$ provides an upper bound on $\hat{\gamma}_{wv}$ and, consequently, (25) implies that $\hat{\gamma}_{wv}\tau < 1$, which proves, using Lemma 2, that system $(\hat{\Sigma}_1, \Sigma_2)$ is $\mathcal{L}_2$ stable with respect to the input/output pair (u, y). Now, using the fact that system $\hat{\Sigma}_1$ is an asymptotically stable linear time-invariant system, Σ_2 has a finite impulse response and the feedback interconnection $(\hat{\Sigma}_1, \Sigma_2)$ satisfies a small gain condition, we can conclude that $(\hat{\Sigma}_1, \Sigma_2)$ is also asymptotically stable for $u = 0$ (see also [25, 41]).

Statement (2): By using the fact that (25) implies the satisfaction of Assumption 2 (note that $\epsilon_{wv} \geq 0$), we can employ (8) in the proof of Lemma 2 to formulate a bound on $\|w\|_2$. Subsequently using (8) and Lemma 1, we can construct the following bound on $\|v\|_2$:

$$\|v\|_2 \leq \frac{\tau\gamma_{wu}}{1-\gamma_{wv}\tau}\|u\|_2. \tag{27}$$

The reduction error on w, defined by $\delta w := w - \hat{w}$, satisfies $\delta w = \mathcal{F}_w(u, v) - \hat{\mathcal{F}}_w(u, \hat{v}) = \mathcal{F}_w(u, v) - \hat{\mathcal{F}}_w(u, v) + \hat{\mathcal{F}}_w(u, v) - \hat{\mathcal{F}}_w(u, \hat{v})$, such that δw can be bounded as follows:

$$\|\delta w\|_2 \leq \|\mathcal{F}_w(u,v) - \hat{\mathcal{F}}_w(u,v)\|_2 + \|\hat{\mathcal{F}}_w(u,v) - \hat{\mathcal{F}}_w(u,\hat{v})\|_2. \tag{28}$$

Herein, we have that

$$\begin{aligned}\|\mathcal{F}_w(u,v) - \hat{\mathcal{F}}_w(u,v)\|_2 &= \|\mathcal{E}_w(u,v)\|_2 \\ &\leq \epsilon_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2,\end{aligned} \tag{29}$$

which follows from (11). Moreover, we have that

$$\|\hat{\mathcal{F}}_w(u,v) - \hat{\mathcal{F}}_w(u,\hat{v})\|_2 \leq \hat{\gamma}_{wv}\|v - \hat{v}\|_2 = \hat{\gamma}_{wv}\|\delta v\|_2 \tag{30}$$

with $\delta v := v - \hat{v}$. Using (29) and (30) in (28) yields

$$\|\delta w\|_2 \leq \epsilon_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2 + \hat{\gamma}_{wv}\|\delta v\|_2. \tag{31}$$

As shown in the proof of statement (1) of the theorem, we have that $\hat{\gamma}_{wv} \leq \gamma_{wv} + \epsilon_{wv}$. Moreover, Lemma 1 implies that $\|\delta v\|_2 \leq \tau\|\delta w\|_2$. Exploiting these two facts in (31) gives

$$\|\delta w\|_2 \leq \frac{1}{1-(\gamma_{wv}+\epsilon_{wv})\tau}(\epsilon_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2), \tag{32}$$

where the small-gain condition in (25) guarantees the existence of the latter bound. Substituting (27) in (32)

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yields

$$\|\delta w\|_2 \leq \frac{1}{1-(\gamma_{wv}+\epsilon_{wv})\tau}\left(\epsilon_{wu}+\frac{\epsilon_{wv}\tau\gamma_{wu}}{1-\gamma_{wv}\tau}\right)\|u\|_2. \tag{33}$$

We employ Lemma 1 once again to obtain a bound on $\|\delta v\|_2$:

$$\|\delta v\|_2 \leq \frac{\tau}{1-(\gamma_{wv}+\epsilon_{wv})\tau}\left(\epsilon_{wu}+\frac{\epsilon_{wv}\tau\gamma_{wu}}{1-\gamma_{wv}\tau}\right)\|u\|_2. \tag{34}$$

The above bound on δv will be exploited to obtain the final error bound on the output y. Hereto, the output error $\delta y := y - \hat{y}$ is considered: $\delta y = \mathcal{F}_y(u,v) - \hat{\mathcal{F}}_y(u,\hat{v}) = \mathcal{F}_y(u,v) - \hat{\mathcal{F}}_y(u,v) + \hat{\mathcal{F}}_y(u,v) - \hat{\mathcal{F}}_y(u,\hat{v})$, such that δy can be bounded as follows: $\|\delta y\|_2 \leq \|\mathcal{F}_y(u,v) - \hat{\mathcal{F}}_y(u,v)\|_2 + \|\hat{\mathcal{F}}_y(u,v) - \hat{\mathcal{F}}_y(u,\hat{v})\|_2$. Using Assumption 3, the latter inequality yields

$$\|\delta y\|_2 \leq \epsilon_{yu}\|u\|_2 + \epsilon_{yv}\|v\|_2 + \hat{\gamma}_{yv}\|\delta v\|_2. \tag{35}$$

Combining (35) with (34), using (27) and the fact that $\hat{\gamma}_{yv} \leq \gamma_{yv} + \epsilon_{yv}$ gives

$$\begin{aligned}\|\delta y\|_2 \leq &\epsilon_{yu}\|u\|_2 + \epsilon_{yv}\frac{\tau\gamma_{wu}}{1-\gamma_{wv}\tau}\|u\|_2 \\ &+ (\gamma_{yv}+\epsilon_{yv})\frac{\tau}{1-(\gamma_{wv}+\epsilon_{wv})\tau} \\ &\times\left(\epsilon_{wu}+\frac{\epsilon_{wv}\tau\gamma_{wu}}{1-\gamma_{wv}\tau}\right)\|u\|_2,\end{aligned} \tag{36}$$

which confirms the validity of the error bound in (26). □

Theorem 1 employs knowledge on the error bounds ϵ_{ij}, $i \in \{y,w\}$, $j \in \{u,v\}$, for the linear reduced-order system $\hat{\Sigma}_1$, providing bounds on all relevant input-output pairs. However, existing model reduction techniques for linear systems generally provide a single error bound ϵ_{lin}, uniform for all input-output pairs. When this error bound is exploited as $\epsilon_{ij} \leq \epsilon_{lin}$ for $i \in \{y,w\}$, $j \in \{u,v\}$, the error bound (26) reduces to

$$\epsilon = \epsilon_{lin}\left(1+\frac{\tau\gamma_{wu}}{1-\gamma_{wv}\tau}\right)\left(1+\frac{(\gamma_{yv}+\epsilon_{lin})\tau}{1-(\gamma_{wv}+\epsilon_{lin})\tau}\right). \tag{37}$$

The small-gain condition in (25) and the error bound (26) only require knowledge on, firstly, properties of the high-order system Σ_1, secondly, the error bound on the linear reduced-order system $\hat{\Sigma}_1$ and, thirdly, the delay and can therefore be evaluated a priori (i.e. without actually performing the reduction first). However, a tighter error bound can be obtained when the gains $\hat{\gamma}_{wv}$ and γ_{yv} of the reduced-order linear subsystem are computed a posteriori (i.e. after the reduction has been employed). These gains can directly be used in (31) and (35), respectively, instead of using their bounds $\gamma_{iv}+\epsilon_{iv}$, $i \in \{y,w\}$. Moreover, the knowledge on $\hat{\gamma}_{wv}$ can be used for the direct evaluation of the small-gain condition via $\hat{\gamma}_{wv}\tau < 1$ instead of via (25), leading to less conservative results.

Remark 5
The results presented above can be extended to a class of nonlinear systems with (potentially uncertain) time-varying delays of the form:

$$\Sigma_{nl}: \begin{cases} \dot{x}(t) = A_0x(t) + B_vf(z(t) - z(t-\tau-\delta\tau(t))) \\ \qquad\quad + Bu(t), \\ z(t) = C_zx(t), \\ y(t) = C_yx(t) + D_{yu}u(t) \end{cases} \tag{38}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^q$, $f: \mathbb{R}^q \to \mathbb{R}^q$, $y \in \mathbb{R}^m$ and $u \in \mathbb{R}^p$, and typically $q \ll n$. Namely, system (38) can indeed be written as a feedback interconnection $(\Sigma_1, \Sigma_{2,nl})$ with Σ_1 as in (3) and $\Sigma_{2,nl}$ given by

$$\Sigma_{2,nl}: v(t) = f\left(\int_{t-\tau-\delta\tau(t)}^{t} w(s)ds\right). \tag{39}$$

If 1) the function f is globally Lipschitz with Lipschitz constant L and 2) the time-varying delay $\tau + \delta\tau(t)$ is a measurable function and satisfies the condition $-\tau \leq -\mu \leq \delta\tau(t) \leq \mu$ for some $\mu \geq 0$ and for all $t \geq 0$, it can be shown (using results in [33], [39]) that the operator $\Sigma_{2,nl}$ satisfies the following incremental gain property: $\|v_2 - v_1\|_2 \leq L\sigma\|w_2 - w_1\|_2$, for all w_1, w_2, with $\sigma := \left(\sqrt{\frac{7}{4}}\mu + \tau\right)$.

Now, under the assumption that the feedback interconnection $(\Sigma_1, \Sigma_{2,nl})$ satisfies the small-gain condition $\gamma_{wv}L\sigma < 1$, extensions of Lemma 2 and Theorem 1 can be obtained, where in the latter τ should be replaced by $L\sigma$.

Systems of the form (38) are common in application fields such as high-speed milling [1, 12, 26] and deep drilling [17, 18] and (without the nonlinearity) also in the scope of networked control systems.

5 Illustrative Example

In order to illustrate the model reduction approach for delay differential equations discussed in Section 3 and the results on the preservation of stability and the error bound in Section 4, we consider the vibration isolation problem of a clamped flexible beam system as depicted

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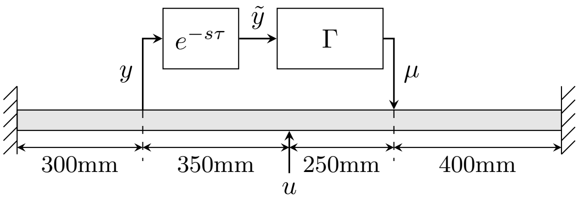

Figure 3. Vibration isolation problem for a flexible beam system.

in Figure 3. The slender beam has the following dimensions: length × height × width = 1.3 m × 3 mm × 0.1 m. Moreover, the beam material properties are as follows: a mass density of 7746 km/m^3 and a Young's modulus of 200 GPa. Moreover, the beam is subject to a disturbance u representing an external force, which causes the beam to vibrate in the vertical plane. To attenuate the effect of these disturbances, an actuation force μ can be applied by a controller, which acts on a measurement $\tilde{y}$ of the vertical deflection y at some point of the beam, see Figure 3. The locations of the disturbance, actuation and sensor are indicated in Figure 3. The dynamics of the beam is modelled using Euler beam elements, leading to a linear time-invariant dynamical system Σ_{beam} of the form

$$M\ddot{q} + D\dot{q} + Kq = b_\mu \mu + b_u u, \quad y = cq \tag{40}$$

with nodal coordinates $q \in \mathbb{R}^N$ and where M, D, and K represent the mass matrix, damping matrix, and stiffness matrix, respectively. For the beam system (40), we assume that the measurement of the vertical deflection y induces a delay, such that the measurement is given as $\tilde{y}(t) = y(t - \tau)$. Then, after transforming (40) to first-order form, an $\mathcal{H}_\infty$-approach is taken to design a linear time-invariant controller Γ that minimizes (in the $\mathcal{H}_\infty$ sense) the transfer function from the external disturbance u to the vertical deflection y by exploiting the actuation force μ and measurements $\tilde{y}$ of the deflection y. In this design procedure, it is assumed that the measurement induces no delay (i.e., $\tau = 0$ and hence $\tilde{y} = y$), such that standard controller synthesis techniques can be applied. Moreover, as a result of the $\mathcal{H}_\infty$ design procedure, the controller is of the same order as Σ_{beam}, such that the closed-loop system has order $n = 4N = 300$.

However, in the analysis of the implemented controller, the measurement delay τ cannot be neglected, such that the closed-loop system is given as in Figure 3. Thus, the resulting closed-loop system Σ comprises a linear delay differential equation [5] that can be written in the form (2), with $n = 300$. Herein, we take the real vertical deflection y (rather than the measurement $\tilde{y}$) as an output of the closed-loop system.

The performance of the controller can be evaluated by means of simulations. However, to reduce the computational burden of such closed-loop performance analysis, the reduction of the closed-loop system Σ is of interest. We stress that the focus of this example is on facilitating numerical simulations and that the individual reduction of the controller (e.g., to enable implementation) is out of the scope of this paper. Nonetheless, we remark that controller reduction can be achieved in a similar setting by exploiting results from [8].

Before discussing the procedure to obtain the reduced-order closed-loop model, we note that the satisfaction of Assumption 1 is guaranteed by the $\mathcal{H}_\infty$ controller design. Namely, the delay is not taken into account in this procedure, guaranteeing asymptotic stability of Σ_1 in (3). Next, by setting the value of the delay to $\tau = 1 \cdot 10^{-2}$s and computing the value of the gain γ_{wv} of Σ_1 as $\gamma_{wv} = 46.70$, we readily check that Assumption 2 is fulfilled. As a result, by Lemma 2, the closed-loop system Σ is $\mathcal{L}_2$ gain stable (from disturbance u to measurement output y) and is asymptotically stable for $u = 0$.

Following the approach of Section 3, a reduced-order model is obtained by applying balanced residualization to obtain a reduced-order model for Σ_1 as $\hat{\Sigma}_1$ (see (10)) of order $\hat{n} = 12$, where we remark that this reduction procedure guarantees the satisfaction of Assumption 3. Moreover, the use of balanced residualization ensures that the infinite-dimensional system resulting from the interconnection of $\hat{\Sigma}_1$ and the delay Σ_2 in (4) can be formulated in terms of a delay differential equation of the form (16), as guaranteed by Proposition 1. More specifically, the reduction procedure is performed on a scaled version of Σ_1, where the signal v is scaled with a factor $S_v = \frac{1}{10}$, such that the small-gain condition of Assumption 2 reads $(\gamma_{wv} S_v)(\tau/S_v) < 1$. The introduction of this scaling allows for balancing the influence of the different outputs of Σ_1, leading to a more accurate reduced-order model. For this scaled model, the error bound as in Assumption 3 is computed as $\epsilon_{wv} = 0.737$ and it readily follows that $((\gamma_{wv} S_v) + \epsilon_{wv})(\tau/S_v) < 1$, such that condition (25) in Theorem 1 is satisfied. As a result, the reduced-order system $(\hat{\Sigma}_1, \Sigma_2)$ is $\mathcal{L}_2$ gain stable from input u to output $\hat{y}$ and is asymptotically stable for $u = 0$. Moreover, the error bound (26) holds, which can be computed as $\epsilon = 33.20$.

Finally, we compare the reduced-order closed-loop system $\hat{\Sigma}$ and the original high-order system Σ by means of their frequency response functions, see Figure 4. Clearly, the reduced-order model provides a good approximation, where we recall that the use of balanced residualization guarantees the preservation of steady-state behavior (or, stated differently, moment matching at $s = 0$, see also Remark 4). Moreover, the uncontrolled system Σ_{beam}

[5] The closed-loop model has been made available in numerical form at the webpage http://twr.cs.kuleuven.be/research/software/delay-control/mor/.

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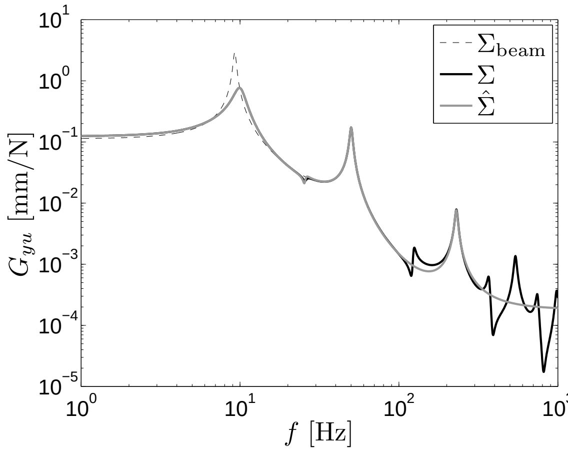

Figure 4. Frequency response function from disturbance u to output y for the open-loop system $\Sigma_{\rm beam}$, the high-order closed-loop system Σ with $n = 300$ and the reduced-order closed-loop system $\hat{\Sigma}$ of order $\hat{n} = 12$.

is depicted in Figure 4, showing the effectiveness of the controller in suppressing the first resonance peak.

6 Conclusions

We have proposed a structure-preserving model reduction approach for a class of delay differential equations. In this approach, a finite-dimensional part of the system is separated from the delay characteristics and the former part is reduced through balancing-type techniques. Benefits of this approach are, firstly, the fact that the delay nature of the system is preserved after reduction, secondly, that input-output stability properties are preserved and, thirdly, that a computable error bound reflecting the accuracy of the reduction is provided. These results are applicable to large-scale linear delay differential equations with constant delays, but also extensions to a class of nonlinear delay differential equations with time-varying delays are presented. The effectiveness of the results is evidenced by means of an illustrative example of a controlled mechanical system with delay in the feedback loop.

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